Non-associative algebras of minimal cones and axial algebras

Vladimir G. Tkachev (Linköping University)

There are remarkable connections of axial algebras to global geometry and regularity theory of nonlinear PDEs. The first example comes from the global geometry of minimal cones. The understanding of geometric and algebraic structure of minimal varieties is a challenging problem with various physical implications ranging from classical general relativity and brane physics [1]. First examples of minimal cones of degree 3 or higher were constructed by Hsiang [2] using the invariance theory. Hsiang also proposed to find a unified explanation for the first nontrivial case: cubic minimal cones. Some progress in the classification was achieved by using representation theory of Clifford algebras [4]. In general, it can be shown [3], [5] that any cubic minimal cone carries a commutative *non-associative algebra structure carrying a bilinear associating form.* The corresponding algebra A is called a *Hsiang algebra.* The correspondence between the analytic and algebraic sides of the problem is as follows. Let $P(x)$ be the defining polynomial of the minimal cone in $\mathbb{R}^n$. Then

- the idempotents in the algebra A are exactly the stationary points of the restriction $P|_S$, S being the unit sphere in $\mathbb{R}^n$;
- the fusion rules come from the minimal surface equation $\operatorname{div}(|\nabla P|^{-1}\nabla P) = \theta|x|^2 P(x)$;
- any element $x \in A$ satisfies $\langle x^2; x^3\rangle = \langle x^2; x\rangle\langle x; x\rangle$ and $\operatorname{trace}\operatorname{ad}(x) = 0$

Any Hsiang algebra A is an axial algebra (spanned by idempotents), where *all idempotents have the same length* 1 *and the same Peirce spectrum* $\sigma(A) = \{-1, -\frac{1}{2}, \frac{1}{2}, 1\}$ satisfying the fusion rules in Table 1. More precisely, the Peirce subspace $A_c(1)$ is 1-dimensional, and $\dim A_c(\lambda)$ does not depend on a choice of an idempotent c, for any $\lambda \in \{-1, -\frac{1}{2}, \frac{1}{2}, 1\}$.

$\star$	1	-1	$-\frac{1}{2}$	$\frac{1}{2}$
1	1	-1	$-\frac{1}{2}$	$\frac{1}{2}$
-1		1	$\frac{1}{2}$	$-\frac{1}{2}, \frac{1}{2}$
$-\frac{1}{2}$			$1, -\frac{1}{2}$	$-1, \frac{1}{2}$
$\frac{1}{2}$				$1, -1, -\frac{1}{2}$

Table 1. Fusion rules for idempotents in Hsiang algebras

The Hsiang algebra fusion rules are no $\mathbb{Z}/2$-graded (as, for instance, for Majorana algebras) but the subspaces $A_1 \oplus A_{-1}$ and $A_1 \oplus A_{-\frac{1}{2}}$ are subalgebras of A. A key classification result states that these subalgebras have hidden Clifford and Jordan algebra structures. An important role in the classification play 2-nilpotent elements and the corresponding Peirce decomposition (see the fusion rules in Table 2). This completely determines the ambient algebra structure.

$\star$	$0'$	-1	1
$0'$	$-1, 1$	$0'', 1$	$0'', -1$
-1		$0'$	$0''$
1			$0'$

Table 2. Fusion rules for a 2-nilpotent w in Hsiang algebras (where $0 = 0' \oplus 0''$, $0' = \langle w\rangle$)

Despite considerable recent progress, the full classification remains open. I will talk about some recent results on algebras of minimal cones. I also relate the latter to general axial algebras and explain why the eigenvalues -1, $-\frac{1}{2}$ and $\frac{1}{2}$ have certain distinguished properties (in the general context of axial algebras).

References

[1] G.W. Gibbons, K.-I. Maeda, and U. Miyamoto. The Bernstein conjecture, minimal cones and critical dimensions. *Classical Quantum Gravity*, 26(18):185008, 14, 2009.

[2] W.-Y. Hsiang. Remarks on closed minimal submanifolds in the standard Riemannian m-sphere. *J. Differential Geometry*, 1:257–267, 1967.

[3] N. Nadirashvili, V.G. Tkachev, and S. Vlăduţ. *Nonlinear elliptic equations and nonassociative algebras*, volume 200 of *Math. Surveys and Monographs.* AMS, Providence, RI, 2014.

[4] V.G. Tkachev. Minimal cubic cones via Clifford algebras. *Complex Anal. Oper. Theory*, 4(3):685–700, 2010.

[5] V.G. Tkachev. Hsiang algebras and cubic minimal cones. *Unpublished manuscript, 140 p.*, 2016.